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On the CP^1 topological sigma model and the Toda lattice hierarchy

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Abstract

We propose, in bihamiltonian formalism, a version of the Toda lattice hierarchy that is satisfied by the two point correlation functions of the CP^1 topological sigma model at genus one approximation, and we also show that this bihamiltonian hierarchy is compatible with the Virasoro constraints of Eguchi–Hori–Xiong up to genus two approximation. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider a 2D topological field theory (TFT) obtained by the coupling of a matter sector to topological gravity. We denote by $\phi_1 = 1, \phi_2, \ldots, \phi_n$ the primary fields of the matter sector, and by $\tau_p(\phi_\alpha)$, $p \geq 1, \alpha = 1, \ldots, n$ with $\tau_0(\phi_\alpha) = \phi_\alpha$ their gravitational descendents, the corresponding coupling constants are denoted by $t^{\alpha,p}$. Then in the genus expansion form the free energy $\mathcal{F}(t)$ of the 2D TFT is expressed as

$$\mathcal{F}(t) = \sum_{g\geq 0} \varepsilon^{2g-2}\mathcal{F}_g(t)$$

with the genus g free energy $\mathcal{F}_g$ defined as the generating function of the genus g correlators

$$\mathcal{F}_g(t) = \langle e^{\sum t^{\alpha,p}\tau_p(\phi_\alpha)}\rangle_g.$$

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If, we restrict the genus zero free energy to the small phase space

$$t^{\alpha,0} = v^{\alpha}, \qquad t^{\alpha,p} = 0, \quad p \geq 1,$$

we get the primary free energy $F(v)$ of the 2D TFT. The primary free energy satisfies a remarkable system of differential equations — the WDVV associativity equations [2,26] which is the base of the theory of Frobenius manifold [4,6]. From the theory of Frobenius manifold, we know that the genus zero free energy $\mathcal{F}_0$ of a 2D TFT can be reconstructed from its primary free energy [5,6], the procedure of reconstruction is described by a bihamiltonian hierarchy of integrable systems called the genus zero bihamiltonian hierarchy, and the genus zero free energy is a particular τ function of this hierarchy. For a 2D TFT with all massive perturbations it has been shown that the genus one free energy can also be reconstructed from the primary free energy through certain deformation of the genus zero bihamiltonian hierarchy [7]. These facts strongly supports our belief that the full genera free energy of any 2D TFT with all massive perturbations should also be a special τ function of certain bihamiltonian hierarchy of integrable systems. This nice picture is realized for the case of pure gravity by the theory of Kontsevich [20] and Witten [27,28], in this case the hierarchy of integrable systems is the well known KdV hierarchy. For a 2D TFT with nontrivial matter sector, however, the hypothetical integrable hierarchy is unknown, the only general result regarding these hierarchies is the explicit expression of their genus one approximation [7]. Nonetheless, there are some conjectures regarding the form of the full genera integrable hierarchy for some particular 2D TFT such as the topological minimal models and the CP^1 topological sigma model [2,4,10–12]. The relation between the CP^1 topological sigma model and bihamiltonian hierarchy of integrable systems is the subject of this note.

The integrable hierarchy that should control the CP^1 topological sigma model was conjectured in [4] to be the Toda lattice hierarchy. In [10–12], it was proved that the conjecture is valid at the genus zero approximation for an appropriate version of the Toda lattice hierarchy, and attempts were also made to check the validity of the conjecture at the genus one and genus two approximation by considering the deformation of the genus zero hierarchy based on the condition of commutativity among the deformed flows. However, since this commutativity condition does not determine the flows uniquely, the genus two approximated flows given in [11] that is supposed to be satisfied by the two point correlation functions of the CP^1 topological sigma model is not proper already in the genus one term, this corresponds to the missing of the Getzler function in the genus one free energy [7,17]. On the other hand, the recursion relations among the flows of the Toda lattice hierarchy that is represented in Lax pair formalism in [11] are not manifestly given, and the appearance of the logarithm of an operator in the Lax pairs makes it already quite nontrivial to give an approximated form of the first set of flows of the hierarchy. The purpose of this note is to propose, in a more explicit form, a version of the Toda lattice hierarchy that is expected to be satisfied by the two point correlation functions of the CP^1 topological sigma model. This version of the Toda lattice hierarchy is expressed in bihamiltonian formalism, and the recursion relations among the flows of the hierarchy are manifestly given by the bihamitonian structure. At the genus one approximation, this hierarchy is verified to be satisfied by the two point correlation functions of the CP^1 topological sigma model, and at the genus two approximation, we prove that this bihamiltonian hierarchy is compatible with the Virasoro constraints that is conjectured to be valid in [14] by Eguchi et al., the precise meaning of this compatibility will be clear in Section 2, where we first recall the genus one approximated bihamiltonian hierarchy that is satisfied by the two point correlation functions of the CP^1 topological sigma model, and then give the genus two correction of the hierarchy by assuming the validity of the Virasoro conjecture [14] at the genus two approximation. In Section 3, we present a version of the bihamiltonian Toda lattice hierarchy, and in Section 4 we define a Miura transformation that establishes the relation between the dynamical variables of the Toda lattice hierarchy and the two point correlation functions of the CP^1 topological sigma model, and show that at the genus two approximation (i.e., up to ε^4) the bihamiltonian Toda lattice hierarchy coincides with the bihamiltonian hierarchy introduced in Section 2 for the CP^1 topological sigma model.

2. The genus two approximated bihamiltonian hierarchy for the CP^1 topological sigma model

In the notation of the last section, the primary free energy of the CP^1 topological sigma model [3,26] has the expression

$$F = \tfrac{1}{2}(v^1)^2 v^2 + \mathrm{e}^{v^2}.$$

The genus zero free energy is a special τ function of a bihamiltonian hierarchy of hydrodynamic type integrable systems, there is a general procedure to construct this hierarchy starting from any solution of the WDVV associativity equations (or a Frobenius manifold) [6], the bihamiltonian structure is defined on the loop space of the Frobenius manifold, and the Hamiltonians of the hierarchy are defined by the flat coordinates of a deformed flat connection of the Frobenius manifold. For our special case of the CP^1 topological sigma model, the bihamiltonian structure is given by

$$\begin{aligned}
&\{v^1(x), v^1(y)\}_1 = \{v^2(x), v^2(y)\}_1 = 0, \qquad \{v^1(x), v^2(y)\}_1 = \delta'(x-y),\\
&\{v^1(x), v^1(y)\}_2 = 2\,\mathrm{e}^{v^2(x)}\delta'(x-y) + v_x^2\,\mathrm{e}^{v^2(x)}\delta(x-y),\\
&\{v^1(x), v^2(y)\}_2 = v^1(x)\delta'(x-y), \qquad \{v^2(x), v^2(y)\}_2 = 2\delta'(x-y).
\end{aligned} \tag{2.1}$$

Let us denote by $(\eta^{\alpha\beta})$ the inverse of the matrix $(\eta_{\alpha\beta})$ with elements

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta},$$

and denote

$$c^{\xi}_{\gamma\nu} = \eta^{\xi\sigma}\frac{\partial^3 F}{\partial v^\gamma \partial v^\nu \partial v^\sigma},$$

here and henceforth summation over the repeated indices is assumed. Then, the Hamiltonians

$$H_{\beta,q} = \int \theta_{\beta,q+1}(v(x))\,\mathrm{d}x, \quad \alpha = 1, 2,\ \ q \geq -1$$

are defined by the relations

$$\theta_{1,0}=v^2,\qquad \theta_{2,0}=v^1,\qquad \frac{\partial^2\theta_{\beta,q+1}}{\partial v^\gamma\partial v^\nu}=c^\xi_{\gamma\nu}\frac{\partial\theta_{\beta,q}}{\partial v^\xi},\quad \alpha,\beta=1,2,\ \ q\ge 0,$$
$$\partial_E\theta_{\beta,q}=(q+\tfrac{1}{2}+\mu_\beta)\theta_{\beta,q}+2\delta_{\beta,1}\theta_{2,q-1}, \tag{2.2}$$

where $\mu_1=-\frac{1}{2}$, $\mu_2=\frac{1}{2}$ and the Euler vector field E is given by

$$E=v^1\frac{\partial}{\partial v^1}+2\frac{\partial}{\partial v^2}.$$

For example, we have

$$\theta_{1,1}=v^1v^2,\qquad \theta_{2,1}=\tfrac{1}{2}(v^1)^2+\mathrm{e}^{v^2},\qquad \theta_{1,2}=\tfrac{1}{2}(v^1)^2v^2+v^2\,\mathrm{e}^{v^2}-2\,\mathrm{e}^{v^2},$$
$$\theta_{2,2}=\tfrac{1}{6}(v^1)^3+v^1\,\mathrm{e}^{v^2}. \tag{2.3}$$

The genus zero bihamiltonian hierarchy for the CP^1 topological sigma model is defined to be

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}}=\{v^\alpha(x),H_{\beta,q}\}_1,\quad \alpha,\beta=1,2,\ \ q\ge 0, \tag{2.4}$$

since $(\partial v^\alpha/\partial t^{1,0})=\partial_x v^\alpha$, we identify the time variable $t^{1,0}$ with the spatial variable x. This hierarchy satisfies the following bihamiltonian recursion relations [6,7,13]

$$\{v^\alpha(x),H_{\beta,q-1}\}_2=(q+\mu_\beta+\tfrac{1}{2})\{v^\alpha(x),H_{\beta,q}\}_1+R^\gamma_\beta\{v^\alpha(x),H_{\gamma,q-1}\}_1,$$
$$\alpha,\beta=1,2,\ \ q\ge 0, \tag{2.5}$$

here

$$R^1_1=R^1_2=R^2_2=0,\qquad R^2_1=2. \tag{2.6}$$

When $\beta=1$, the above recursion procedure starts from $q=1$.

The genus zero two point correlation functions

$$\eta^{\alpha\gamma}\frac{\partial^2\mathcal{F}_0(t)}{\partial t^{1,0}\partial t^{\gamma,0}},\quad \alpha=1,2 \tag{2.7}$$

give a particular solution $v^{(0)}(t)$ of the above hierarchy, this solution is specified by the initial condition

$$v^{(0)^\alpha}(t)|_{t^{\beta,q\ge 1}=0}=t^{\alpha,0} \tag{2.8}$$

and by the symmetry reduction

$$\frac{\partial v^{(0)}}{\partial t^{1,1}}-\sum_{\alpha,p}t^{\alpha,p}\frac{\partial v^{(0)}}{\partial t^{\alpha,p}}=0 \tag{2.9}$$

which leads, in particular, to the string equation.

Now, let us look at the genus one approximation of the CP^1 topological sigma model. The genus one free energy is given by [3,7,17]

$$\mathcal{F}_1(t) = F_1(v, v_x)|_{v=v^{(0)}(t),\, v_x=\partial_x v^{(0)}(t)}$$

with

$$F_1 = \tfrac{1}{24}\log\det(c^{\alpha}_{\beta\gamma}\, v^{\gamma}_x) - \tfrac{1}{24}v^2,$$

it was shown in [7] that the genus one two point correlation functions

$$\frac{\partial^2\mathcal{F}_0(t)}{\partial t^{1,0}\partial t^{\alpha,0}} + \varepsilon^2\frac{\partial^2\mathcal{F}_1(t)}{\partial t^{1,0}\partial t^{\alpha,0}} + \mathcal{O}(\varepsilon^4)$$

satisfy a bihamiltonian hierarchy, which is a deformation of the genus zero hierarchy (2.4) obtained by the following transformation:

$$u^{\alpha} = v^{\alpha} + \varepsilon^2\eta^{\alpha\gamma}\frac{\partial^2 F_1(v, v_x)}{\partial t^{1,0}\partial t^{\gamma,0}}. \tag{2.10}$$

The deformed hierarchy has the bihamiltonian structure

$$\{u^{\alpha}(x), u^{\beta}(y)\}_i = \{u^{\alpha}(x), u^{\beta}(y)\}^{(0)}_i + \varepsilon^2\{u^{\alpha}(x), u^{\beta}(y)\}^{(1)}_i + \mathcal{O}(\varepsilon^4), \tag{2.11}$$

where $\{u^{\alpha}(x), u^{\beta}(y)\}^{(0)}_i$ are defined by (2.1) with v^1, v^2 replaced by u^1, u^2, and

$$\begin{aligned}
&\{u^1(x), u^1(y)\}^{(1)}_1 = \{u^2(x), u^2(y)\}^{(1)}_1 = 0, \qquad \{u^1(x), u^2(y)\}^{(1)}_1 = -\tfrac{1}{12}\delta'''(x-y),\\
&\{u^1(x), u^1(y)\}^{(1)}_2 = \mathrm{e}^{u^2(x)}(\tfrac{1}{6}\delta'''(x-y) + \tfrac{1}{4}u^2_x\delta''(x-y) + (\tfrac{1}{12}(u^2_x(x))^2\\
&\qquad\qquad + \tfrac{1}{4}u^2_{xx})\delta'(x-y) + (\tfrac{1}{12}u^2_x\, u^2_{xx} + \tfrac{1}{12}u^2_{xxx})\delta(x-y)),\\
&\{u^1(x), u^2(y)\}^{(1)}_2 = -\tfrac{1}{12}u^1(x)\delta'''(x-y) - \tfrac{1}{12}u^1_x\,\delta''(x-y),\\
&\{u^2(x), u^2(y)\}^{(1)}_2 = 0.
\end{aligned} \tag{2.12}$$

We note here the remarkable property of the deformed bihamiltonian structure (2.11) and (2.12) that it is given by differential polynomials in $u^{\alpha}_x, u^{\alpha}_{xx}, \ldots$, this property is rather nontrivial because our Miura type transformation (2.10) is given by rational instead of polynomial functions in $v^{\alpha}_x, v^{\alpha}_{xx}, \ldots$. We call such transformations *quasi-Miura transformations* in [9].

The Hamiltonians of the deformed hierarchy

$$\frac{\partial u^{\alpha}}{\partial t^{\beta,q}} = \{u^{\alpha}(x), H_{\beta,q}\}_1 \tag{2.13}$$

is obtained from the genus zero ones by transforming the v^{α} coordinates to the u^{α} coordinates, the densities of the Hamiltonians can be chosen as polynomials in the x-derivatives of the new coordinates. An explicit formula of such a choice was given in [7] for a general semisimple Frobenius manifold. For the CP^1 topological sigma model we have, e.g.,

$$\begin{aligned} H_{\beta,-1} &= \int \eta_{\beta\gamma} u^\gamma(x)\,\mathrm{d}x + \mathcal{O}(\varepsilon^4), \quad \beta = 1, 2, \\ H_{1,0} &= \int \theta_{1,1}(u(x))\,\mathrm{d}x - \varepsilon^2 \int \frac{1}{12} u_x^1(x) u_x^2(x)\,\mathrm{d}x + \mathcal{O}(\varepsilon^4), \\ H_{2,0} &= \int \theta_{2,1}(u(x))\,\mathrm{d}x - \varepsilon^2 \int \left(\frac{1}{24}(u_x^1)^2 + \frac{1}{12}(u_x^2)^2\,\mathrm{e}^{u^2(x)}\right)\mathrm{d}x + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{2.14}$$

Let us consider now the genus two approximation of the CP^1 topological sigma model. It was conjectured in [14] that the partition function of a topological sigma model should be annihilated by an infinite set of Virasoro operators, this conjecture was proved to be valid at the genus zero approximation [8,22], and for any topological sigma model with semisimple quantum cohomology this conjecture was also proved to be true at genus one approximation [8,23]. In particular, the genus zero and genus one free energy of the CP^1 topological sigma model satisfy the genus one Virasoro constraints. It was also conjectured in [15] that, as for the genus one free energy, the higher genera free energy could be expressed as a function of the genus zero correlation functions in the form

$$\mathcal{F}_g(t) = F_g(v, v_x, \ldots, \partial_x^{3g-2} v)|_{v=v^{(0)}(t)}. \tag{2.15}$$

In [9], it was shown that the genus two Virasoro constraints lead to a unique solution F_2 of the above form for any generic two-dimensional Frobenius manifold, and the explicit expression of F_2 were given there. A similar derivation of the genus two free energy for two-dimensional Frobenius manifold is also given in [16] by using the above form of $\mathcal{F}_2(t)$, the genus two Virasoro constraints and the genus two topological recursion relations [1,18]. To write down the explicit formula of the function F_2, let us denote

$$V_{\alpha_1,\ldots,\alpha_k} = \eta_{\alpha_1\gamma} \frac{\partial^{k-1} v^\gamma}{\partial t^{\alpha_2,0} \cdots \partial t^{\alpha_k,0}}, \quad k = 1, 2, \ldots,$$

$$\begin{aligned} Q_1 &= V_{1,\alpha_1,\alpha_2,\alpha_3,\alpha_4} (M^{-1})^{\alpha_1\alpha_2} (M^{-1})^{\alpha_3\alpha_4}, \\ Q_2 &= V_{1,\alpha_1,\alpha_2,\alpha_3} V_{\alpha_4,\alpha_5,\alpha_6} (M^{-1})^{\alpha_1\alpha_4} (M^{-1})^{\alpha_2\alpha_5} (M^{-1})^{\alpha_3\alpha_6}, \\ Q_3 &= V_{1,\alpha_1,\alpha_2} V_{\alpha_3,\alpha_4,\alpha_5,\alpha_6} (M^{-1})^{\alpha_1\alpha_3} (M^{-1})^{\alpha_2\alpha_4} (M^{-1})^{\alpha_5\alpha_6}, \\ Q_4 &= V_{1,\alpha_1,\alpha_2} V_{\alpha_3,\alpha_4,\alpha_5} V_{\alpha_6,\alpha_7,\alpha_8} (M^{-1})^{\alpha_1\alpha_3} (M^{-1})^{\alpha_2\alpha_6} (M^{-1})^{\alpha_4\alpha_7} (M^{-1})^{\alpha_5\alpha_8}, \\ Q_5 &= V_{1,\alpha_1,\alpha_2} \frac{\partial^2 G}{\partial t^{\alpha_3,0} \partial t^{\alpha_4,0}} (M^{-1})^{\alpha_1\alpha_3} (M^{-1})^{\alpha_2\alpha_4}, \\ Q_6 &= \frac{\partial^3 G}{\partial t^{1,0} \partial t^{\alpha_1,0} \partial t^{\alpha_2,0}} (M^{-1})^{\alpha_1\alpha_2}, \end{aligned}$$

where

$$(M^{-1})^{\alpha\beta} = (M^{-1})^\alpha_\gamma\, \eta^{\gamma\beta}, \qquad (M^\alpha_\beta) = (c^\alpha_{\beta\gamma} v_x^\gamma)$$

and

$$G = -\tfrac{1}{24} v^2.$$

With these notations, the genus two free energy for the CP^1 topological sigma model is given by

$$F_2 = \tfrac{1}{1152} Q_1 - \tfrac{1}{360} Q_2 - \tfrac{1}{1152} Q_3 + \tfrac{1}{360} Q_4 - \tfrac{11}{240} Q_5 + \tfrac{1}{20} Q_6 + \tfrac{7}{5760} v_{xx}^2.$$

Being a unique solution of the form (2.15) to the genus two Virasoro constraints, $\mathcal{F}_2$ also satisfies the genus two topological recursion relations given in [1,18]. In [9], it was proved that, at the approximation up to ε^4, the two point correlation functions

$$\frac{\partial^2 \mathcal{F}_0(t)}{\partial t^{1,0} \partial t^{\alpha,0}} + \varepsilon^2 \frac{\partial^2 \mathcal{F}_1(t)}{\partial t^{1,0} \partial t^{\alpha,0}} + \varepsilon^4 \frac{\partial^2 \mathcal{F}_2(t)}{\partial t^{1,0} \partial t^{\alpha,0}} + \mathcal{O}(\varepsilon^6)$$

also satisfy a bihamiltonian hierarchy of integrable systems, which is a deformation of the genus zero one (2.4), under the following quasi-Miura transformation:

$$u^\alpha = v^\alpha + \varepsilon^2 \eta^{\alpha\gamma} \frac{\partial^2 F_1(v, v_x)}{\partial t^{1,0} \partial t^{\gamma,0}} + \varepsilon^4 \eta^{\alpha\gamma} \frac{\partial^2 F_2(v, v_x, \ldots, \partial_x^4 v)}{\partial t^{1,0} \partial t^{\gamma,0}}. \tag{2.16}$$

The bihamiltonian structure is given by

$$\begin{aligned} \{u^\alpha(x), u^\beta(y)\}_i = {} & \{u^\alpha(x), u^\beta(y)\}_i^{(0)} + \varepsilon^2 \{u^\alpha(x), u^\beta(y)\}_i^{(1)} \\ & + \varepsilon^4 \{u^\alpha(x), u^\beta(y)\}_i^{(2)} + \mathcal{O}(\varepsilon^6), \end{aligned} \tag{2.17}$$

where $\{u^\alpha(x), u^\beta(y)\}_i^{(0)}$, $\{u^\alpha(x), u^\beta(y)\}_i^{(1)}$ are defined as in (2.11) and (2.12), and

$$\begin{aligned} & \{u^1(x), u^1(y)\}_1^{(2)} = \{u^2(x), u^2(y)\}_1^{(2)} = 0, \qquad \{u^1(x), u^2(y)\}_1^{(2)} = \tfrac{1}{240} \delta^{(5)}, \\ & \{u^1(x), u^1(y)\}_2^{(2)} = \mathrm{e}^{u^2(x)} \Big[-\tfrac{1}{360} \delta^{(5)} - \tfrac{1}{144} u_x^2 \delta^{(4)} + \tfrac{1}{180} u_{xx}^2 \delta''' - \tfrac{1}{120} (u_x^2)^2 \delta''' \\ & \quad + \tfrac{11}{720} u_{xxx}^2 \delta'' + \tfrac{1}{240} u_{xx}^2 u_x^2 \delta'' - \tfrac{1}{180} (u_x^2)^3 \delta'' + \tfrac{1}{90} \partial_x^4 u^2 \delta' \\ & \quad + \tfrac{1}{120} u_{xxx}^2 u_x^2 \delta' + \tfrac{7}{720} (u_{xx}^2)^2 \delta' - \tfrac{1}{720} u_{xx}^2 (u_x^2)^2 \delta' - \tfrac{1}{720} (u_x^2)^4 \delta' \\ & \quad + \left(\tfrac{1}{288} u_x^2 (u_{xx}^2)^2 + \tfrac{1}{360} u_x^2 \partial_x^4 u^2 + \tfrac{1}{144} u_{xx}^2 u_{xxx}^2 + \tfrac{1}{360} \partial_x^5 u^2 \right) \delta \Big], \\ & \{u^1(x), u^2(y)\}_2^{(2)} = \tfrac{1}{240} u^1(x) \delta^{(5)} + \tfrac{1}{120} u_x^1 \delta^{(4)} + \tfrac{1}{180} u_{xx}^1 \delta''' + \tfrac{1}{720} u_{xxx}^1 \delta'', \\ & \{u^2(x), u^2(y)\}_2^{(2)} = -\tfrac{1}{120} \delta^{(5)}, \end{aligned} \tag{2.18}$$

here we omitted, to save spaces, the arguments x–y of the δ function and its derivatives.

We remark here again that the quasi-Miura transformation (2.16) is given by complicated rational functions in the x-derivatives of the v^α variables, however, the resulting deformation of the bihamiltonian structure is polynomial in the x-derivatives of the new dependent variables u^α (at the approximation up to ε^4). In [9], we call such deformation of the genus zero bihamiltonian structure (2.1) a quasi-trivial deformation, general properties of such transformations are studied there.

To describe the genus two deformation of the hierarchy (2.4), we also need to express the Hamiltonians $H_{\beta,q}$ in terms of the new coordinates u^α, we have

$$\begin{aligned} H_{\beta,-1} &= \int \eta_{\beta\gamma} u^\gamma(x)\,\mathrm{d}x + \mathcal{O}(\varepsilon^6),\\ H_{1,0} &= \int u^1(x)u^2(x)\,\mathrm{d}x - \varepsilon^2 \int \frac{1}{12} u^1_x(x) u^2_x(x)\,\mathrm{d}x - \frac{\varepsilon^4}{360}\int u^1_x u^2_{xxx}\,\mathrm{d}x + \mathcal{O}(\varepsilon^6),\\ H_{2,0} &= \int \left(\frac{1}{2}(u^1(x))^2 + \mathrm{e}^{u^2(x)}\right)\mathrm{d}x - \varepsilon^2 \int \left(\frac{1}{24}(u^1_x)^2 + \frac{1}{12}(u^2_x)^2\,\mathrm{e}^{u^2(x)}\right)\mathrm{d}x\\ &\quad + \varepsilon^4 \int \left(\frac{1}{720}(u^1_{xx})^2 + \frac{1}{160}(u^2_{xx})^2\,\mathrm{e}^{u^2(x)} + \frac{1}{360} u^2_{xx}(u^2_x)^2\,\mathrm{e}^{u^2(x)}\right)\mathrm{d}x + \mathcal{O}(\varepsilon^6). \end{aligned} \tag{2.19}$$

Due to (2.5), these Hamiltonians satisfy the following recursion relations:

$$\begin{aligned} &\{u^\alpha(x), H_{\beta,q-1}\}_2 = (q + \mu_\beta + \tfrac{1}{2})\{u^\alpha(x), H_{\beta,q}\}_1 + R^\gamma_\beta \{u^\alpha(x), H_{\gamma,q-1}\}_1,\\ &\alpha, \beta = 1, 2, \quad q \ge 0, \end{aligned} \tag{2.20}$$

these recursion relations can be solved to yield a unique set of Hamiltonians $H_{\beta,q}$, whose densities can be expressed in the form

$$\tilde{\theta}_{\beta,q+1} = \theta_{\beta,q+1}(u) + \varepsilon^2 \theta^{(1)}_{\beta,q+1}(u, u_x) + \varepsilon^4 \theta^{(2)}_{\beta,q+1}(u, u_x, \dots, \partial_x^4 u) + \mathcal{O}(\varepsilon^6),$$

where $\theta_{\beta,q}(u)$ are given by (2.2), and $\theta^{(1)}_{\beta,q+1}(u, u_x), \theta^{(2)}_{\beta,q+1}(u, u_x, \dots, \partial_x^4 u)$ are polynomials in the x-derivatives of u^1, u^2. This is guaranteed by the vanishing of the Poisson cohomologies for the Poisson structure

$$\{u^\alpha(x), u^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y) \tag{2.21}$$

proved in [9,19] and also independently by Magri. The resulting bihamiltonian hierarchy is then expressed as

$$\frac{\partial u^\alpha}{\partial t^{\beta,q}} = \{u^\alpha(x), H_{\beta,q}\}_1, \quad \alpha, \beta = 1, 2, \quad q \ge 0, \tag{2.22}$$

where the Poisson brackets are defined in (2.17) and (2.18). The right-hand side of the systems (2.22) have the form

$$K^{(0)}_{\alpha;\beta,q}(u, u_x) + \varepsilon^2 K^{(1)}_{\alpha;\beta,q}(u, u_x, u_{xx}, u_{xxx}) + \varepsilon^4 K^{(2)}_{\alpha;\beta,q}(u, u_x, \dots, \partial_x^5 u) + \mathcal{O}(\varepsilon^6),$$

where $K^{(l)}_{\alpha;\beta,q}$ are polynomials in the x-derivatives of u^α. From our construction we see that, at the approximation up to ε^2, this hierarchy coincides with the hierarchy that is satisfied by the two point correlation functions of the CP^1 topological sigma model at genus one approximation, it is also clear now what is the meaning of its compatibility with the genus two Virasoro constraints.

In [7,9], it was conjectured that the bihamiltonian structure of the hypothetical integrable hierarchy for a 2D TFT with all massive perturbations should be a quasi-trivial deformation of that of the genus zero bihamiltonian hierarchy, and the bihamiltonian structure has the form

$$\begin{aligned}\{u^\alpha(x), u^\beta(y)\}_i &= \{u^\alpha(x), u^\beta(y)\}_i^{(0)} \\ &\quad + \sum_{l\geq 1}\sum_{s=0}^{2l+1} \varepsilon^{2l} P^{\alpha\beta}_{i;l,s}(u; u_x, u_{xx}, \dots, \partial_x^s u)\delta^{(2l+1-s)}(x-y), \\ &\quad i = 1, 2,\end{aligned} \tag{2.23}$$

where $P^{\alpha\beta}_{i;l,s}(u; u_x, u_{xx}, \dots, \partial_x^s u)$ are differential polynomials in $u^\alpha_x, u^\alpha_{xx}, \dots$, and the total degree of the x-derivatives equals s. In the next two sections, we will present a version of the Toda lattice hierarchy whose bihamiltonian structure meets the above form and coincides with that of (2.17) and (2.18) up to ε^4.

3. The Toda lattice hierarchy

The Toda lattice equation has the form [25]

$$\frac{\partial^2 q_n}{\partial t^2} = \mathrm{e}^{q_{n-1}-q_n} - \mathrm{e}^{q_n - q_{n+1}}, \quad n \in \mathbf{Z}, \tag{3.1}$$

if we denote

$$g_n^1 = -\frac{\partial q_n}{\partial t}, \qquad g_n^2 = q_{n-1} - q_n,$$

then the Toda lattice Eq. (3.1) can be put into the form

$$\frac{\partial g_n^1}{\partial t} = \mathrm{e}^{g_{n+1}^2} - \mathrm{e}^{g_n^2}, \qquad \frac{\partial g_n^2}{\partial t} = g_n^1 - g_{n-1}^1, \quad n \in \mathbf{Z}. \tag{3.2}$$

The above system has a bihamiltonian structure which can be found, e.g., in [21]. By introducing the slow variables $t^{2,0} = \varepsilon t$, $x = \varepsilon n$ and the new dependent variables

$$g^1(x) = g_n^1, \qquad g^2(x) = g_n^2,$$

we can rewrite the system (3.2) in the following form:

$$\frac{\partial g^1}{\partial t^{2,0}} = \frac{1}{\varepsilon}(\mathrm{e}^{g^2(x+\varepsilon)} - \mathrm{e}^{g^2(x)}), \qquad \frac{\partial g^2}{\partial t^{2,0}} = \frac{1}{\varepsilon}(g^1(x) - g^1(x-\varepsilon)). \tag{3.3}$$

The reason to use $t^{2,0}$ as the time variable for the above system will be clear in the next section, we are also to use $t^{\beta,q}$ as the time variables for the Toda lattice hierarchy which will be defined soon. In our notation, the bihamiltonian structure for (3.3) has the form [7,21]

$$\begin{aligned}
&\{g^1(x), g^1(y)\}_1 = \{g^2(x), g^2(y)\}_2 = 0,\\
&\{g^1(x), g^2(y)\}_1 = \frac{1}{\varepsilon}(\delta(x-y+\varepsilon) - \delta(x-y)),\\
&\{g^1(x), g^1(y)\}_2 = \frac{1}{\varepsilon}(\mathrm{e}^{g^2(x+\varepsilon)}\delta(x-y+\varepsilon) - \mathrm{e}^{g^2(x)}\delta(x-y-\varepsilon)),\\
&\{g^1(x), g^2(y)\}_2 = \frac{1}{\varepsilon}g^1(x)(\delta(x-y+\varepsilon) - \delta(x-y)),\\
&\{g^2(x), g^2(y)\}_2 = \frac{1}{\varepsilon}(\delta(x-y+\varepsilon) - \delta(x-y-\varepsilon)). \qquad (3.4)
\end{aligned}$$

The above bihamiltonian structure can be expanded in power series of ε, and the coefficients of ε^0 give precisely the genus zero bihamiltonian structure (2.1). In the next section, we will perform a Miura transformation to convert it into the form of (2.23).

The first Poisson structure has two Casimirs

$$h_{1,-1} = \int g^2(x)\,\mathrm{d}x, \qquad h_{2,-1} = \int g^1(x)\,\mathrm{d}x. \qquad (3.5)$$

We now proceed to define the Toda lattice hierarchy by using the recursion relations

$$\begin{aligned}
&\{g^\alpha(x), h_{\beta,q-1}\}_2 = (q + \mu_\beta + \tfrac{1}{2})\{g^\alpha(x), h_{\beta,q}\}_1 + R^\gamma_\beta\{g^\alpha(x), h_{\gamma,q-1}\}_1,\\
&\qquad \alpha, \beta = 1, 2, \quad q \geq 0, \qquad (3.6)
\end{aligned}$$

where R is define in (2.6). From the Casimir $h_{2,-1}$ the above recursion relations generate one branch of our Toda lattice hierarchy with Hamiltonians $h_{2,q}$, this branch of the hierarchy is usually referred to as the Toda lattice hierarchy in the literature (see, e.g., [21] and the remark in the end of this section). The first flow

$$\frac{\partial g^\alpha}{\partial t^{2,0}} = \{g^\alpha(x), h_{2,0}\}_1$$

with Hamiltonian

$$h_{2,0} = \int \left[\frac{1}{2}(g^1)^2(x) + \mathrm{e}^{g^2(x)}\right]\mathrm{d}x \qquad (3.7)$$

is just the Toda lattice equations (3.3). Since $\{g^\alpha(x), h_{1,-1}\}_2 = 0$, we no longer able to start from the Casimir $h_{1,-1}$ to generate the second branch of the hierarchy, instead, we have to start the recursion procedure from the Hamiltonian $h_{1,0}$. Before giving the definition of $h_{1,0}$, let us introduce some notations which will be used in the definition of the second branch of the Toda lattice hierarchy and also in the next section. Denote by A, B the shift operators

$$A(x;\varepsilon)f(x) := f(x+\varepsilon) - f(x) = \sum_{k\geq 1}\frac{\varepsilon^k}{k!}\partial_x^k f(x), \qquad (3.8)$$

$$\begin{aligned}
B(x;\varepsilon)f(x) &:= -A(x;\varepsilon)A(x;-\varepsilon)f(x) = -A(x;-\varepsilon)A(x;\varepsilon)f(x)\\
&= f(x+\varepsilon) + f(x-\varepsilon) - 2f(x) = \sum_{k\geq 1}\frac{2\varepsilon^{2k}}{(2k)!}\partial_x^{2k} f(x) \qquad (3.9)
\end{aligned}$$

for an arbitrary smooth function $f(x)$. In the notation of these two operators, we indicate explicitly the variable x to which the shift operation takes place, this is because we will apply these operators on functions with more then one variables later, for example, when the operator $A(y,\varepsilon)$ is applied to the function $f(x,y)$, we get

$$A(y,\varepsilon)f(x,y)=f(x,y+\varepsilon)-f(x,y)=\sum_{k\geq 1}\frac{\varepsilon^k}{k!}\partial_y^k f(x,y).$$

We specify the inverse of the operators A and B by

$$A^{-1}(x;\varepsilon)\partial_x f(x)=\frac{1}{\varepsilon}\sum_{k\geq 0}a_k\varepsilon^k\partial_x^k f(x), \tag{3.10}$$

$$B^{-1}(x;\varepsilon)\partial_x^2 f(x)=\frac{1}{\varepsilon^2}\sum_{k\geq 1}b_k\varepsilon^{2k}\partial_x^{2k} f(x), \tag{3.11}$$

where the coefficients a_k, b_k are defined by

$$a_0=1,\qquad a_1=-\frac{1}{2},\qquad a_k=-\sum_{l=1}^{k}\frac{a_{k-l}}{(l+1)!},\qquad b_0=1,\qquad b_1=-\frac{1}{12},$$
$$b_k=-\sum_{l=1}^{k}\frac{2b_{k-l}}{(2l+2)!},\quad k\geq 2. \tag{3.12}$$

The coefficients a_k have the property

$$a_{2l+1}=0,\quad l\geq 1, \tag{3.13}$$

this can be seen from the identity

$$\varepsilon A^{-1}(x;\varepsilon)\partial_x f(x)-(-\varepsilon)A^{-1}(x,-\varepsilon)\partial_x f(x)=2\sum_{l\geq 0}a_{2l+1}\varepsilon^{2l+1}\partial_x^{2l+1}f(x),$$

since the left-hand side of the above identity equals

$$-\varepsilon B^{-1}(x;\varepsilon)(A(x;\varepsilon)\partial_x f+A(x;-\varepsilon)\partial_x f)=-\varepsilon B^{-1}(x;\varepsilon)B(x,\varepsilon)\partial_x f(x)=-\varepsilon\partial_x f.$$

Now, we define the Hamiltonian

$$h_{1,0}=\int \varepsilon g^2(x)\,A^{-1}(x;\varepsilon)\partial_x g^1(x)\,\mathrm{d}x=\int g^2(x)\sum_{k\geq 0}a_k\varepsilon^k\partial_x^k g^1(x)\,\mathrm{d}x, \tag{3.14}$$

and define the Hamiltonians $h_{\beta,q}$, $\beta=1,2$, $q\geq 1$ by using the recursion relation (3.6). The existence of these Hamiltonians is again ensured by the vanishing of the Poisson cohomologies of the Poisson structure (2.21), their densities can be represented by differential polynomials in the x-derivatives of g^1, g^2 and can be expanded in a power series of ε. These Hamiltonians are uniquely determined by the condition that in the density of $h_{\beta,q}$, after being expanded in power series of ε, the leading term coincides with $\theta_{\beta,q}$ that is defined in (2.2). We can require this condition because when expanded as power series of ε, the leading terms of the bihamiltonian structure (3.4) coincide with the bihamiltonian structure (2.1). For example, we have

$$\begin{aligned} h_{1,1} &= \int \left[\mathrm{e}^{g^2(x)} \sum_{k\geq 0} (-\varepsilon)^k a_k \partial_x^k g^2(x) + \frac{1}{2} g^2(x) \sum_{k\geq 0} \varepsilon^k a_k \partial_x^k (g^1(x))^2 \right. \\ &\qquad \left. + g^1(x) \sum_{k\geq 0} \varepsilon^k a_k \partial_x^k g^1(x) \right] \mathrm{d}x - 2h_{2,0}, \\ h_{2,1} &= \int \left[\frac{1}{6} (g^1(x))^3 + \frac{1}{2} g^1(x) (\mathrm{e}^{g^2(x+\varepsilon)} + \mathrm{e}^{g^2(x)}) \right] \mathrm{d}x. \end{aligned} \tag{3.15}$$

We define the flows of the Toda lattice hierarchy by the following systems:

$$\frac{\partial g^\alpha}{\partial t^{\beta,q}} = \{g^\alpha(x), h_{\beta,q}\}_1, \quad \alpha, \beta = 1, 2, \quad q \geq 0, \tag{3.16}$$

where the Poisson bracket is defined in (3.4). The $t^{2,0}$ flow is just the Toda lattice equations (3.3), and the $t^{1,0}$ flow is the shift along x, i.e.

$$\frac{\partial g^\alpha}{\partial t^{1,0}} = \partial_x g^\alpha.$$

Proposition 3.1. *The flows of the Toda lattice hierarchy commute with each other*.

Proof. We use Magri's standard procedure [24] to prove the commutativity of the flows of the Toda lattice hierarchy. By using the recursion relations (3.6), we have

$$\{h_{2,p}, h_{2,q}\}_1 = \binom{p+q+2}{q+1} \{h_{2,p+q+1}, h_{2,-1}\}_1 = 0, \tag{3.17}$$

here the last equality is due to the fact that $h_{2,-1}$ is a Casimir of the first Poisson bracket. Similarly, by using (3.6) and (3.17), we get

$$\{h_{1,p}, h_{2,q}\}_1 = \binom{p+q+1}{p} \{h_{1,p+q+1}, h_{2,-1}\}_1 = 0, \tag{3.18}$$

by using this identity and again the recursion relation (3.6), we get

$$\begin{aligned} \binom{p+q}{q}^{-1} \{h_{1,p}, h_{1,q}\}_1 = \{h_{1,p+q}, h_{1,0}\}_1 &= \int \frac{\delta h_{1,p+q}}{\delta g^\gamma(x)} \{g^\gamma(x), h_{1,0}\}_1 \, \mathrm{d}x \\ &= \int \frac{\delta h_{1,p+q}}{\delta g^\gamma(x)} \partial_x g^\gamma(x) \, \mathrm{d}x = 0. \end{aligned} \tag{3.19}$$

The commutativity of the flows (3.16) follows immediately from the Jacobi identity for the first Poisson bracket and the commutativity of the Hamiltonians. □

Remark. As we already mentioned, the branch of the Toda lattice hierarchy (3.16) that consists of the systems with flows $\partial/\partial t^{2,q}$, $q \geq 0$ is usually called the Toda lattice hierarchy in the literature [21], in the discrete form these systems have the following Lax representations:

$$\frac{\partial L}{\partial \tilde{t}^{2,p}} = \frac{1}{(p+1)!}[(L^{p+1})_+, L], \quad p \geq 0,$$

where the time variables $\tilde{t}^{2,p}$ is related to $t^{2,p}$ through $t^{2,p} = \varepsilon \tilde{t}^{2,p}$, and the Lax operator L is given by

$$L = \Lambda + g_n^1 + \mathrm{e}^{g_n^2} \Lambda^{-1}$$

with Λ being the shift operator that acts on the discrete variable n, i.e.,

$$\Lambda f_n = f_{n+1},$$

and the operator $(L^{p+1})_+$ is obtained from L^{p+1} by dropping those terms in L^{p+1} with negative powers of Λ, e.g., we have $(L)_+ = \Lambda + g_n^1$.

4. A Miura transformation relating the Toda lattice hierarchy to the CP^1 topological sigma model

Define the following Miura transformation:

$$\begin{aligned} w^1(x) &= \varepsilon A^{-1}(x;\varepsilon)\partial_x g^1(x) = \sum_{k\geq 0} a_k \varepsilon^k \partial_x^k g^1(x), \\ w^2(x) &= \varepsilon^2 B^{-1}(x;\varepsilon)\partial_x^2 g^2(x) = \sum_{k\geq 0} b_k \varepsilon^{2k} \partial_x^{2k} g^2(x), \end{aligned} \tag{4.1}$$

where a_k, b_k are defined in (3.12). We now transform the bihamiltonian structure of the Toda lattice hierarchy (3.16) into these new coordinates to establish its relation to that of the CP^1 topological sigma model given in Section 2.

Proposition 4.1. *In the new coordinates w^1, w^2, the bihamiltonian structure* (3.4) *has the expression*

$$\begin{aligned} \{w^1(x), w^1(y)\}_1 &= \{w^2(x), w^2(y)\}_1 = 0, \\ \{w^1(x), w^2(y)\}_1 &= \sum_{k\geq 0}(1-2k)a_{2k}\varepsilon^{2k}\delta^{(2k+1)}(x-y) = \sum_{k\geq 0} b_k \varepsilon^{2k}\delta^{(2k+1)}(x-y), \\ \{w^1(x), w^1(y)\}_2 &= 2\,\mathrm{e}^{g^2(x)}\delta'(x-y) + \mathrm{e}^{g^2(x)} g_x^2(x)\delta(x-y) \\ &\quad + \sum_{m\geq 1}\varepsilon^{2m} a_{2m}\Bigg[2\,\mathrm{e}^{g^2(x)}\delta^{(2m+1)}(x-y) + (2m+1)\partial_x(\mathrm{e}^{g^2(x)}) \\ &\quad \times \delta^{(2m)}(x-y) + \sum_{l=0}^{2m-2}\binom{2m}{l}\partial_x^{2m-l}(\mathrm{e}^{g^2(x)})\delta^{(l+1)}(x-y)\Bigg], \end{aligned}$$

$$
\begin{aligned}
\{w^1(x), w^2(y)\}_2 &= w^1(x)\delta'(x-y) \\
&\quad + \sum_{k\ge 1} a_{2k}\varepsilon^{2k}\left[w^1(x)\delta^{(2k+1)}(x-y)\right. \\
&\quad \left. - \sum_{l=1}^{2k} \binom{2k}{l} \partial_x^{l-1} w^1(x)\delta^{(2k+2-l)}(x-y)\right], \\
\{w^2(x), w^2(y)\}_2 &= 2\sum_{k,m\ge 0} a_{2k} b_m \varepsilon^{2m+2k}\delta^{(2m+2k+1)}(x-y),
\end{aligned}
\tag{4.2}
$$

where the variable $g^2(x)$ is expressed in terms of w^2 through

$$
g^2 = \sum_{k\ge 0} \frac{2}{(2k+2)!}\varepsilon^{2k}\partial_x^{2k} w^2. \tag{4.3}
$$

Proof. Since the Miura transformation (4.1) is linear, the derivation of the bihamiltonian structure in the new coordinates is straightforward. We show as example the derivation of the formula for $\{w^1(x), w^1(y)\}_2$, by using (4.1) we have

$$
\begin{aligned}
\{w^1(x), w^1(y)\}_2 &= \varepsilon^2 A^{-1}(x;\varepsilon) A^{-1}(y;\varepsilon)\partial_x\partial_y\{g^1(x), g^1(y)\}_2 \\
&= \varepsilon A^{-1}(x;\varepsilon)A^{-1}(y;\varepsilon)\partial_x\partial_y[A(x;\varepsilon)(\mathrm{e}^{g^2(x)}\delta(x-y)) \\
&\quad - \mathrm{e}^{g^2(x)}A(y,\varepsilon)\delta(x-y)] \\
&= \varepsilon A^{-1}(y;\varepsilon)\partial_x[\mathrm{e}^{g^2(x)}\partial_y\delta(x-y)] - \varepsilon A^{-1}(x;\varepsilon)\partial_x[\mathrm{e}^{g^2(x)}\partial_y\delta(x-y)] \\
&= \partial_x\left[\sum_{k\ge 0}(-1)^k a_k\varepsilon^k\,\mathrm{e}^{g^2(x)}\delta^{(k)}(x-y)\right] + \sum_{k\ge 0} a_k\varepsilon^k\partial_x^k[\mathrm{e}^{g^2(x)}\delta'(x-y)] \\
&= 2\,\mathrm{e}^{g^2(x)}\delta'(x-y) + \mathrm{e}^{g^2(x)} g_x^2(x)\delta(x-y) + \sum_{m\ge 1}\varepsilon^{2m} a_{2m} \\
&\quad \times\left[2\,\mathrm{e}^{g^2(x)}\delta^{(2m+1)}(x-y) + (2m+1)\partial_x(\mathrm{e}^{g^2(x)})\delta^{(2m)}(x-y)\right. \\
&\quad \left. + \sum_{l=0}^{2m-2}\binom{2m}{l}\partial_x^{2m-l}(\mathrm{e}^{g^2(x)})\delta^{(l+1)}(x-y)\right],
\end{aligned}
$$

here the last equality holds true due to (3.13). □

By using the above proposition, a simple calculation yields the following.

Proposition 4.2. *After substituting g^2 by* (4.3) *and expanding into power series of ε, the bihamiltonian structure* (4.2) *has the form* (2.23) *and, moreover, by identifying w^α with u^α, this bihamiltonian structure coincides with the bihamiltonian structure given in* (2.17) and (2.18) *for the CP^1 topological sigma model at the approximation up to ε^4.*

Under the Miura transformation (4.1), the Toda lattice hierarchy (3.16) is transformed to

$$
\begin{aligned}
\frac{\partial w^1}{\partial t^{\beta,q}} &= \{w^1(x), h_{\beta,q}\}_1 = \frac{\partial}{\partial x}\frac{\delta h_{\beta,q}}{\delta g^2},\\
\frac{\partial w^2}{\partial t^{\beta,q}} &= \{w^2(x), h_{\beta,q}\}_1 = \frac{\partial}{\partial x}\sum_{k\ge 0} a_k \varepsilon^k \partial_x^k \frac{\delta h_{\beta,q}}{\delta g^1},
\end{aligned} \tag{4.4}
$$

where the Poisson bracket is given in (4.2), $h_{\beta,q}$ are defined in Section 3, and the right-hand side of the above equations can be easily represented by the new coordinates w^α through the substitution of (4.3) and of

$$
g^1 = \sum_{k\ge 0} \frac{\varepsilon^k}{(k+1)!} \partial_x^k w^1.
$$

In the new coordinates, the Hamiltonians $h_{\beta,q}$ satisfy the recursion relation

$$
\begin{aligned}
&\{w^\alpha(x), h_{\beta,q-1}\}_2 = (q + \mu_\beta + \tfrac{1}{2})\{w^\alpha(x), h_{\beta,q}\}_1 + R_\beta^\gamma \{w^\alpha(x), h_{\gamma,q-1}\}_1,\\
&\alpha, \beta = 1, 2,\quad q \ge 0,
\end{aligned} \tag{4.5}
$$

which have the same form of (2.20) if we identify w^α with u^α, so due to the above proposition and our definition of the Hamiltonians $h_{\beta,q}$ in the last section, we have the following corollary.

Corollary 4.1. *Under the identification of w^α with u^α, the Toda lattice hierarchy* (4.4) *coincides with the bihamiltonian hierarchy* (2.22) *satisfied by the two point approximation of the CP^1 topological sigma model at the genus two approximation* (*i.e., at the approximation up to ε^4*).

Let us write down in the coordinates w^α the first two sets of flows of the Toda lattice hierarchy:

$$
\begin{aligned}
\frac{\partial w^\alpha}{\partial t^{1,0}} &= \partial_x w^\alpha, \qquad \frac{\partial w^2}{\partial t^{2,0}} = \partial_x w^1,\\
\frac{\partial w^1}{\partial t^{2,0}} &= \partial_x\, \mathrm{e}^{g^2} = \partial_x \exp\left(\sum_{k\ge 0} \frac{2}{(2k+2)!} \varepsilon^{2k} \partial_x^{2k} w^2\right),\\
\frac{\partial w^1}{\partial t^{1,1}} &= \partial_x \left(\sum_{k\ge 0} \varepsilon^{2k} a_{2k} (\partial_x^{2k}\, \mathrm{e}^{g^2} + \mathrm{e}^{g^2} \partial_x^{2k} g^2) + \tfrac{1}{2}\mathcal{P}(w^1, w^1) - 2\,\mathrm{e}^{g^2}\right),\\
\frac{\partial w^2}{\partial t^{1,1}} &= \partial_x \left(\mathcal{P}(w^1, w^2) + 2\sum_{k\ge 1} \varepsilon^{2k} a_{2k} \partial_x^{2k} w^1\right),
\end{aligned}
$$

$$\frac{\partial w^1}{\partial t^{2,1}} = \partial_x \left(e^{g^2} \sum_{k\geq 0} \frac{\varepsilon^{2k}}{(2k+1)!} \partial_x^{2k} w^1 \right),$$
$$\frac{\partial w^2}{\partial t^{2,1}} = \partial_x \left(\frac{1}{2} \mathcal{P}(w^1, w^1) + \sum_{k\geq 0} \varepsilon^{2k} a_{2k} \partial_x^{2k} e^{g^2} \right), \tag{4.6}$$

where

$$\mathcal{P}(w^\alpha, w^\beta) = \sum_{k\geq 0} \sum_{m=0}^{2k} \sum_{l=0}^{2k-m} a_{2k-m-l} \varepsilon^{2k} \partial_x^{2k-m-l} \left(\frac{(\partial_x^m w^\alpha)(\partial_x^l w^\beta)}{(m+1)!(l+1)!} \right).$$

and g^2 is related to w^2 through (4.3).

5. Concluding remarks

We have shown that, at the approximation up to ε^4, the bihamiltonian Toda lattice hierarchy (4.4) coincides with the bihamiltonian hierarchy that is satisfied by the two point correlation functions of the CP^1 topological sigma model at genus two approximation, here the genus two approximation of the CP^1 topological sigma model is obtained by assuming the validity of the genus two Virasoro constraints and the conjecture on the form of the genus two free energy (2.15). To prove this relation between the Toda lattice hierarchy and the CP^1 topological sigma model, we show that at the approximation up to ε^4 the bihamiltonian structure (4.2) of the Toda lattice hierarchy is in fact a quasi-trivial deformation of the genus zero bihamiltonian structure (2.1). We believe that this quasi-triviality property of the bihamitonian structure (4.2) of the Toda lattice hierarchy is also valid in full genera, to say more explicitly, there should exist a quasi-Miura transformation of the form

$$w^\alpha = v^\alpha + \eta^{\alpha\gamma} \sum_{g\geq 1} \varepsilon^{2g} \frac{\partial^2 F_g(v, \partial_x v, \ldots, \partial_x^{3g-2} v)}{\partial t^{1,0} \partial t^{\gamma,0}}$$

such that the genus zero bihamiltonian structure (2.1) is transformed to (4.2), here the genus zero flows $\partial v^\alpha / \partial t^{\beta,q}$ are needed in the expression of the quasi-Miura transformation. This quasi-triviality property is in fact quite strong, we believe that it determines the quasi-Miura transformation (thus all the functions F_g) uniquely, and if this is realized, then the genus g free energy of the CP^1 topological sigma model could be calculated from the knowledge of the genus zero correlation functions through

$$\mathcal{F}_g(t) = F_g(v, \partial_x v, \ldots, \partial_x^{3g-2} v)|_{\partial_x^k v = \partial_x^k v^{(0)}(t)},$$

where $v^{(0)}(t)$ is given by the genus zero correlation functions (2.7). The notion of quasi-trivial deformations of a bihamiltonian structure is introduced and its important properties are given in [9], examples of quasi-trivial deformations of the genus zero bihamiltonian structures defined on the loop space of any generic two-dimensional Frobenius manifold are given at the genus two approximation there, these quasi-trivial deformations are proved to be compatible with the genus two Virasoro constraints.

Finally, we remark that in [7] a linear Miura transformation that is different from that of (4.1) was given, although it does transform the bihamiltonian structure (3.4)–(4.2) at the approximation up to ε^3, it is in fact not able to be corrected to become a linear Miura transformation between the two bihamiltonian structures at the approximation up to ε^4.

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References

[1] P. Belorosski, R. Pandharipande, A descendent relation in genus 2, math/9803072.
[2] R. Dijkgraaf, E. Verlinde, H. Verlinde, Topological strings in $d < 1$, Nucl. Phys. B 352 (1991) 59.
[3] R. Dijkgraaf, E. Witten, Mean field theory, topological field theory, and multi-matrix models, Nucl. Phys. B 342 (1990) 486–522.
[4] B. Dubrovin, Integrable systems and classification of 2-dimensional topological field theories, in: O. Babelon, O. Cartier, Y. Kosmann-Schwarbach (Eds.), Integrable Systems, Proceedings of Luminy 1991 Conference Dedicated to the Memory of J.-L. Verdier, Birkhäuser, 1993.
[5] B. Dubrovin, Integrable systems in topological field theory, Nucl. Phys. B 379 (1992) 627–689.
[6] B. Dubrovin, Geometry of 2D topological field theories, in: M. Francaviglia, S. Greco (Eds.), Integrable Systems and Quantum Groups, Montecalini, Terme, 1993, Springer Lecture Notes in Mathematics, Vol. 1620, Springer, Berlin, 1996, pp. 120–348.
[7] B. Dubrovin, Y. Zhang, Bihamiltonian hierarchies in 2D topological field theory at one loop approximation, Commun. Math. Phys. 198 (1998) 311–361.
[8] B. Dubrovin, Y. Zhang, Frobenius manifolds and Virasoro constraints, Selecta Math. New Ser. 5 (1999) 423–466.
[9] B. Dubrovin, Y. Zhang, Normal forms of integrable hierarchies and Gromov–Witten invariants, in press.
[10] T. Eguchi, H. Kanno, Toda lattice hierarchy and the topological description of the $c = 1$ string theory, Phys. Lett. B 331 (1994) 330–334.
[11] T. Eguchi, S.-K. Yang, The topological CP^1 model and the large-N matrix integral, Mod. Phys. Lett. A 9 (1994) 2893–2902.
[12] T. Eguchi, Y. Yamada, S.-K. Yang, On the genus expansion in the topological string theory, Rev. Math. Phys. 7 (1995) 279.
[13] T. Eguchi, K. Hori, C.S. Xiong, Gravitational quantum cohomology, Int. J. Mod. Phys. A 12 (1997) 1743–1782.
[14] T. Eguchi, K. Hori, C.S. Xiong, Quantum cohomology and Virasoro algebra, Phys. Lett. B 402 (1997) 71–80.
[15] T. Eguchi, C.S. Xiong, Quantum cohomology at higher genus: topological recursion relations and Virasoro conditions, Adv. Theor. Math. Phys. 2 (1998) 219–229.
[16] T. Eguchi, E. Getzler, C.S. Xiong, Topological gravity in genus 2 with two primary fields, hep-th/0007194.
[17] E. Getzler, Intersection theory on $\bar{M}_{1,4}$ and elliptic Gromov–Witten invariants, alg-geom/9612004, J. Am. Math. Soc. 10 (1997) 973–998.
[18] E. Getzler, Topological recursion relations in genus 2, math/9801003.
[19] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, math.DG/0002164.

[20] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. 147 (1992) 1–23.
[21] B.A. Kupershmidt, Discrete Lax equations and differential difference calculus, Astérisque, 1985.
[22] X. Liu, G. Tian, Virasoro constraints for quantum cohomology, J. Diff. Geom. 50 (1998) 537–590.
[23] X. Liu, Elliptic Gromov–Witten invariants and Virasoro conjecture, math.AG/9907113.
[24] F. Magri, A simple model of the integrable Hamiltonian systems, J. Math. Phys. 19 (1978) 1156–1162.
[25] M. Toda, Wave propagation in anharmonic lattices, J. Phys. Soc. Jpn. 23 (1967) 501–506.
[26] E. Witten, On the structure of the topological phase of two-dimensional gravity, Nucl. Phys. B 340 (1990) 281–332.
[27] E. Witten, Two-dimensional gravity and intersection theory on moduli space, Surv. Diff. Geom. 1 (1991) 243–310.
[28] E. Witten, On the Kontsevich model and other models of two-dimensional gravity, Preprint IASSNS-HEP-91/24.